PHASE TRANSITION IN SUPERCONDUCTORS

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It is shown that the phase transition in a superconductor takes place in the same way as in Bose liquids. Cooper pairs play the role of bosons. The specific heat has a logarithmic singularity in a very narrow temperature range.

ACCORDING to contemporary representations, the λ transition in He and the transition into the superconducting state in metals have a common nature and are connected with the Einstein condensation of Bose particles. However, experiment shows that in the transition in He, the heat capacity C_p has a logarithmic singularity, whereas in the superconducting transition C_p undergoes a finite jump.

The theory of the phase transition in a superconductor [1] was developed on the basis of a model of a Hamiltonian [2] in which only the interaction of particles with opposite momenta and spins is taken into account. This model admits of an exact solution and gives a finite jump in the heat capacity. The theory of the phase transition in a Bose liquid [3] shows that the interaction of large dimension fluctuations plays the fundamental role in the origin of the logarithmic singularity in thermodynamic functions. In the Hamiltonian model of [2] such an interaction is excluded. Therefore, to develop a theory of transition to the superconducting state, it is necessary to consider a more realistic model.

We shall consider a Fermi liquid, the transition temperature T_0 of which is small in comparison with the degeneracy temperature μ (or with the Debye temperature in the case of a metal). It will be shown that the picture of the phase transition is the same as for a Bose liquid, and that the Cooper pairs play the role of Bose particles. The region of the logarithmic phase transition to the superconducting state is shown to be very small $(T - T_0)/T_0 \sim (T_0/\mu)^4$. This is connected with the weakness of the pair interaction because of their low density and the low effective mass. Unfortunately, such a small temperature range is not yet accessible to experiment.

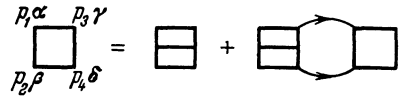
Only temperatures $T \geq T_0$ will be considered here.

1. PROPERTIES OF THE VERTEX PART FOR SMALL TOTAL MOMENTUM

When $T = T_0$, Cooper pairs of electrons are formed with opposite momenta and spins [4] and with zero binding energy. Therefore, the scattering amplitude of two electrons for zero total 4-momentum and spin has a pole at $T = T_0$.

We consider the properties of this vertex for small total momentum and for temperatures close to T_0. Strictly speaking, we consider the vertex part $\Gamma_{\alpha\beta\gamma\delta}(p_1p_2; p_3p_4)$, which depends on discrete frequencies and from which one can obtain the scattering amplitude by analytic continuation.

The graphic equation for the vertex part $\Gamma_{\alpha\beta\gamma\delta}$ has the form



(1.1)

Here the crossed square denotes the set of graphs that are not cut for two electron lines with the same direction. We denote the contribution of these graphs by the expression $\Gamma^{(1)}_{\alpha\beta\gamma\delta}(p_1p_2; p_3p_4)$.

By assumption, the paired electrons have opposite spins. Therefore, the part $\Gamma_{\alpha\beta\gamma\delta}$, which has the singularity, is antisymmetric in the spins and symmetric in the momenta, i.e., it has the form

$$\Gamma(p_1p_2;\ p_3p_4)\,(\delta_{\alpha\gamma}\,\delta_{\beta\delta} - \delta_{\alpha\delta}\,\delta_{\beta\gamma}), \tag{1.2}$$

where Γ is symmetric relative to the substitution $p_1 \longleftrightarrow p_2$, $p_3 \longleftrightarrow p_4$. We also select the part of $\Gamma^{(1)}_{\alpha\beta\gamma\delta}$ antisymmetric in the spins.

It is convenient to transform to the variable: $q = p_1 + p_2 = p_3 + p_4$, and, for example, p_1, p_3 such that $\Gamma(p_1, p_2; p_3p_4) \equiv \Gamma(p_1, p_3; q)$. In this notation, Eq. (1.1) can be written in the following form:

$$\Gamma(p_1, p_3; q) = \Gamma^{(1)}(p_1, p_3; q) - T\sum_{n}\int\frac{d^3k}{(2\pi)^3}\,\Gamma^{(1)}(p_1k; q)\,G(k)\,G(q-k)\,\Gamma(kp_3; q). \tag{1.3}$$

Here G(k) is the Green's function of the electron.

We shall investigate Eq. (1.3) in the region of small $|q|$ for zero value of the fourth component of q. It is assumed that G(k) has the usual form:[5]

$$G(k) \approx \frac{a}{i\omega_n - v(k-p_0)} \qquad (\omega_n, v|k-p_0| \ll \mu), \tag{1.4}$$

where v is the Fermi velocity, p_0 is the Fermi momentum, a is a constant, and $\omega_n = (2n+1)\pi T$.

As $T \to T_0$ and $q \to 0$, the function $\Gamma^{(1)}(p_1p_3;q)$ tends to the finite limit $\gamma^{(1)}(p_1p_3)$. Here, $\gamma^{(1)}(p_1p_3)$ is regarded as an analytic function of its frequency arguments, coinciding with $\Gamma^{(1)}(p_1p_3;0)$ for $T = T_0$. The limit of G(k) as $T \to T_0$ is denoted by $G_0(k)$.

We now introduce the auxiliary quantity $\gamma(p_1p_3)$, defined by the equation

$$\gamma(p_1p_3) = \gamma^{(1)}(p_1p_3) - T\sum_n \int \frac{d^3k}{(2\pi)^3}\gamma^{(1)}(p_1k)G_0(k)G_0(-k)\gamma(kp_3). \tag{1.5}$$

The quantities $\bar\gamma^{(1)} = \Gamma^{(1)} - \gamma^{(1)}$ and $GG^- - G_0G_0^-$ ($G^- = G(q-k)$, $G_0^- = G_0(-k)$) are small. Equation (1.3) can be rewritten in the following fashion:

$$\Gamma(p_1p_3;q) = \gamma(p_1p_3) - T\sum_n \int \frac{d^3k}{(2\pi)^3}\gamma(p_1k)[G(k)G(q-k) - G_0(k)G_0(-k)]\Gamma(kp_3;q) + T^2\sum_{nm}\int \frac{d^3p d^3k}{(2\pi)^6}\gamma(p_1p)G_0(p) \times G_0(-p)\bar\gamma^{(1)}(pk;q)G_0(k)G_0(-k)\Gamma(kp_3;q). \tag{1.6}$$

For proof of (1.6) we write down Eq. (1.3) in symbolic operator form:

$$\Gamma = \Gamma^{(1)} + \Gamma^{(1)}GG^-\Gamma. \tag{1.3'}$$

Substituting $\Gamma^{(1)} = \gamma^{(1)} + \bar\gamma^{(1)}$, $\Gamma = \gamma + \bar\gamma$, and $\Gamma^{(1)}GG^- = \gamma^{(1)}G_0G_0^- + \tau$, and using the equation for γ

$$\gamma = \gamma^{(1)} + \gamma^{(1)}G_0G_0^-\gamma, \tag{1.5'}$$

we get an equation for $\bar\gamma$

$$\bar\gamma = \bar\gamma^{(1)} + \gamma^{(1)}G_0G_0^-\bar\gamma + \tau\Gamma. \tag{1.7}$$

The first component in (1.7) can be neglected since it does not have a singularity. For $\bar\gamma$, we find

$$\bar\gamma = (1 - \gamma^{(1)}G_0G_0^-)^{-1}\tau\Gamma. \tag{1.8}$$

It follows from (1.5) that

$$(1 - \gamma^{(1)}G_0G_0^-)^{-1} = \gamma(\gamma^{(1)})^{-1}. \tag{1.9}$$

We are considering the equations (1.3') and (1.5') near the poles of Γ and γ. Therefore $\gamma^{(1)}$ approximately satisfies the equation

$$(\gamma^{(1)})^{-1} = G_0G_0^-. \tag{1.10}$$

Substituting (1.9) and (1.10) in (1.8), we get

$$\bar\gamma = \gamma G_0G_0^-\bar\gamma^{(1)}G_0G_0^-\Gamma + \gamma G_0G_0^-\Gamma^{(1)}[GG^- - G_0G_0^-]\Gamma. \tag{1.11}$$

Substituting $\Gamma^{(1)} \to \gamma^{(1)}$ in the last term and using Eq. (1.10), we get the following expression for $\bar\gamma$

$$\bar\gamma = \gamma G_0G_0^-\bar\gamma^{(1)}G_0G_0^-\Gamma + \gamma[GG^- - G_0G_0^-]\Gamma, \tag{1.12}$$

whence Eq. (1.6) follows.

It is shown in the Appendix that the solution of Eq. (1.5) has the form

$$\gamma(p_1p_3) = -\frac{\chi(p_1)\chi(p_3)}{T - T_0}, \tag{1.13}$$

where $\chi(p)$ satisfies the equation

$$\chi(p) = -T_0\sum_n \int \frac{d^3k}{(2\pi)^3}\gamma^{(1)}(pk)G_0(k)G_0(-k)\chi(k) \tag{1.14}$$

and the normalization condition

$$\chi|_{p=p_0,\omega=0} = (T_0/\alpha)^{1/2}, \qquad \alpha \equiv a_0^2p_0^2/2\pi^2v_0. \tag{1.15}$$

Equation (1.14) is identical with the linearized equation for the gap at the transition point.

From Eqs. (1.7) and (1.13), we get the equation for

$$\Gamma(p_1p_3;q) = -\frac{\chi(p_1)\chi(p_3)}{T - T_0 + \Phi(q)}, \tag{1.16}$$

$$\Phi(q) = -T\sum_n \int \frac{d^3p}{(2\pi)^3}\chi^2(p)[G(p)G(q-p) - G_0(p)G_0(-p)] + T^2\sum_{nm}\int \frac{d^3pd^3k}{(2\pi)^6}\chi(p)G_0(p)G_0(-p) \times \bar\gamma^{(1)}(pk;q)G_0(k)G_0(-k)\chi(k). \tag{1.17}$$

The quantity defined by Eq. (1.16) is essentially the amplitude of resonance scattering of the electron by an electron through a virtual pair. The virtual pair with momentum q can be set in correspondence with the "Green's function"

$$\mathfrak{G}(q) = -(T - T_0 + \Phi(q))^{-1}, \tag{1.18}$$

and the vertex corresponding to the decay (production) of the pair corresponds to the function $\chi(p)$.

The boson Green's function $\mathfrak{G}(q)$ can be represented in the same form as in the theory of a phase transition in a Bose liquid:[3]

$$\mathfrak{G}(q) = -(\eta + \varphi(q))^{-1}; \tag{1.19}$$

$$\eta = T - T_0 + \Phi(0) = T - T_0 + S(0) - S_0(0), \tag{1.20}$$

$$\varphi(q) = \Phi(q) - \Phi(0) = S(q) - S(0), \tag{1.21}$$

where the quantity

$$S(q) = -T\sum_n \int \frac{d^3p}{(2\pi)^3}\chi^2(p)G(p)G(q-p) + T^2\sum_{nm}\int \frac{d^3pd^3k}{(2\pi)^6}\chi(p)G_0(p)G_0(-p)\Gamma^{(1)}(pk;q) \times G_0(k)G_0(-k)\chi(k) \tag{1.22}$$

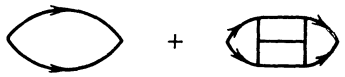
plays the role of the self-energy part of the boson.

The quantity $S_0(0)$ in Eq. (1.20) is obtained from S(q) if we replace G by G_0, $\Gamma^{(1)}$ by $\gamma^{(1)}$ and set q = 0 in (1.22).

We shall associate the function $\mathfrak{G}(q)$ with the heavy line in the graphs.

2. THE EQUATION FOR THE GREEN'S FUNCTION OF A BOSON

The self-energy part of the boson S(q) can be represented graphically in the form

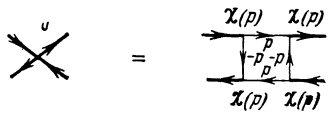


(2.1)

The first term in (2.1) makes a contribution $\sim v^2q^2/T_0$ to $\varphi(q)$. The second term in (2.1) is represented in the form of a set of graphs consisting of the Green's functions of the boson.

The singularity in Γ [see (1.16)] is brought about by the set of graphs cut for two electron lines directed to one side. Thus the thick line essentially describes the set of cut graphs.

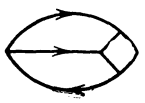
We separate from (2.1) the graphs which do not contain $\mathfrak{G}$ lines. These graphs make the contribution $\sim v^2q^2/\mu$ to $\varphi(q)$, which can be neglected. The graphs in (2.1) which contain $\mathfrak{G}$ lines can be constructed in the following fashion. We introduce the "bare" vertex V_0, which describes the interaction of bosons as a set of graphs with four boson outputs which do not contain interior boson lines. In the case of a weak interaction of electrons, vertex V_0 corresponds to the graph



(2.2)

In this case $V_0 \sim 1/p_0$. Since this estimate does not contain the electron-electron interaction constant g, it remains correct even for a non-weak interaction $gp_0 \sim 1$.

Let us construct all the possible graphs which contain only V_0 and $\mathfrak{G}$ lines and which are not cut for a single $\mathfrak{G}$ line. The set of such graphs can be represented in the following fashion



(2.3)

where the square corresponds to the total 4-pole constructed from V_0 and $\mathfrak{G}$ lines. The set of graphs (2.3) forms a part of (2.1). The other graphs in (2.1) can be constructed by introducing the "bare" 6-, 8-poles and so forth. As has already been shown in [3], these graphs give upon subtraction the higher powers of q in $\varphi(q)$; therefore, the contribution of these graphs is unimportant in the equation for $\varphi(q)$. In contrast with the situation arising in the case of a Bose liquid, [3] these graphs do not lead to renormalization of V_0 because of the smallness of T_0.

One can show that the sign of the graph is determined by the number $(-1)^n$, where n is the number of "bare" vertices just as in the Bose case. Thus the equation for $\varphi(q)$ has the same form as in the case of the Bose liquid [see [3], Eq. (2.9)]. Exactly the same equations for the Bose vertex parts reduce to the corresponding equations of theory of the phase transition in a Bose liquid [see [3], Eqs. (2.10)–(2.12)]. In particular, $\varphi(q)$ has on the transition line the form

$$\varphi(q) = Aq^{3/2}, \tag{2.4}$$

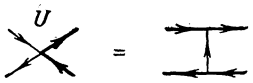
where $A^2 \sim V_0T_0 \sim T_0/p_0$, and the Bose vertex part is a homogeneous function of zero order in the momenta.

For the calculation of the thermodynamic functions, we need an expression for the vertex part of electrons $\Gamma(p_1p_2; p_3p_4)$ with small momentum transfer $p_1 - p_3 = k$ and zero frequency shift $\omega_1 - \omega_3 = 0$. For this purpose we introduce the notation

$$\Gamma(p_1p_2; p_3p_4) = K(p_1p_2; k).$$

For the construction of the vertex part $K(p_1p_2; k)$, we introduce the "bare" vertex U for electron scattering by the pair, i.e., the vertex with two G and two $\mathfrak{G}$ exterior lines, which does not contain $\mathfrak{G}$ lines internally.

In the case of a weak interaction of the electrons, this vertex corresponds to the graph



(2.5)

One can draw the vertex K graphically in the form:

(2.6)

Here the shaded square represents graphs which do not contain $\mathfrak{G}$ lines. The remaining graphs in (2.6) make a small contribution $\sim T_0/\mu$. The first graph in (2.6) contains no singularities. The other two graphs are identical for k = 0 with the graphs for $\partial N_B/\partial \mu_B$, accurate to within a multiplying factor (N_B is the boson density, μ_B is the chemical potential) for the case of a Bose liquid. As was shown in [3], $K \sim \ln \eta$. For $k \neq 0$, $K \sim \ln \max\{\eta, Ak^{3/2}\}$.

3. THE THERMODYNAMICS OF THE PHASE TRANSITION

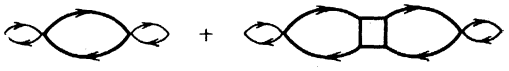
We investigate the thermodynamics of the phase transition in a fashion similar to the case of a Bose liquid.[3] We compute the value of $\partial N_F/\partial\mu$ (N_F is the electron density):

$$\frac{\partial N_F}{\partial\mu} = 2T\sum_n \int \frac{d^3p}{(2\pi)^3}\frac{\partial G(p)}{\partial\mu}. \tag{3.1}$$

We make use of the Ward identity[5]

$$\frac{\partial G(p)}{\partial\mu} = -G^2(p)\left\{1 - 2T\sum_n \int \frac{d^3q}{(2\pi)^3} K(pq;0)\,G^2(q)\right\}. \tag{3.2}$$

In Eq. (3.2) the term K(pq; 0) containing the vertex part with zero transfer yields a logarithmic singularity. The basic term in $\partial N_F/\partial\mu$ can be represented by the graph



(3.3)

Here we shall make use of the graphic equation (2.6) for the vertex part K. It is evident from (3.3) that

$$\partial N_F/\partial\mu = a^2\partial N_B/\partial\mu_B, \tag{3.4}$$

$$a = 2T\sum_n \int \frac{d^3p}{(2\pi)^3} G^2(p)\,U(p), \tag{3.5}$$

a is a quantity of the order of unity. Making use of the formulas[3] for $\partial N_B/\partial\mu_B$,[1)] we get

$$\frac{\partial N_F}{\partial\mu} = a^2\frac{R}{V_0}\ln\frac{\Delta T}{\eta} + R_1, \tag{3.6}$$

where R is a universal, dimensionless constant, $R_1 \sim p_0$ is finite on the transition curve, and ΔT is the interval of the logarithmic phase transition. In order of magnitude, ΔT is equal to the potential energy $Aq^{3/2}$, in which case it is equal to the "kinetic" energy v^2q^2/T_0, i.e.,

$$\Delta T/T_0 \sim (T_0/\mu)^4. \tag{3.7}$$

Simple calculations show that the heat capacity C_V does not contain a logarithmic singularity, while the singular part of C_p has the form

$$C_p = a^2\frac{T_0R}{N_F^2V_0}\left(\frac{\partial p}{\partial T}\right)_V^2\ln\frac{\Delta T}{\eta}. \tag{3.8}$$

The quantity η is essentially linearly dependent on $T - T_0$ and $\mu - \mu_0$. In order of magnitude, $(\partial p/\partial T)_V \sim p_0^3$.

The results are an additional argument for the universal character of phase transitions of the second kind.

In conclusion, we would like to touch upon certain problems arising in the comparison of the consequences of the theory of superconductivity with experiment.

The interval of the logarithmic phase transition, in accord with the estimate of (3.7), is very small. We do not know the numerical factor in this estimate; however, it can scarcely change the order of the quantity ΔT enough to make this effect accessible to experiment. From the results of this work it follows that the thermodynamics of superconductivity in the model of Bardeen, Cooper and Schrieffer are valid to within the interval of the logarithmic phase transition, i.e., in practice, to within T_0. Moreover, the theory, which is based on the weak coupling approximation, should be the better satisfied the closer T is to T_0, inasmuch as in this theory the expansion is actually carried out over the small parameter $(T - T_0)/T_0$.[6] In this connection, it is completely beyond understanding why the inequality $\Delta C/C < 1.4$, introduced within the framework of the theory of weak coupling, is not satisfied. Éliashberg[7] advanced the opinion that the limit of applicability of the BCS theory is determined by the range of temperatures in which the damping becomes of the order of the gap. However, calculations[2)] do not bear out this assumption.

APPENDIX

In Sec. 1, we introduced an auxiliary function $\gamma(p_1p_3)$ defined by the equation

$$\gamma(p_1p_3) = \gamma^{(1)}(p_1p_3) - T\sum_n \int \frac{d^3k}{(2\pi)^3}\gamma^{(1)}(p_1k)\,G_0(k)\,G_0(-k)\,\gamma(kp_3). \tag{A.1}$$

We are interested in the behavior of γ at temperatures close to T_0. The dependence of γ on T can be determined by making use of the fact that G_0 and $\gamma^{(1)}$ do not depend on T, by definition. The temperature enters into G_0 and $\gamma^{(1)}$ only through the discrete frequencies. For real frequencies, G_0 and $\gamma^{(1)}$ do not in general contain the temperature. Therefore, it is convenient to carry out analytic continuation of (A.1) on the real axis. One then obtains for γ, as a function of the temperature, a differential equation which is easily solved.

The transition from summation over discrete frequencies to integration over the real axis is

[1)] In this case no renormalization due to nonzero frequencies takes place, since $\mathfrak{G}$ is defined only for $\omega = 0$.

[2)] T. K. Melik-Barkhudarov, private communication.

effected in the usual fashion with the help of the function $\tanh(\epsilon/2T)$, which has poles at the points $\epsilon = i\omega_n$. Here it is necessary to take into account the analytic properties of the integrands. As is well known, G_0 has a cut along the real axis, while $\gamma^{(1)}(p_1p_3)$ and $\gamma(p_1p_3)$ have the following cuts as analytic functions of the frequency variables ϵ_1 and ϵ_3 (see [8]):

$$1)\ \mathrm{Im}\,\varepsilon_1 = \mathrm{Im}\,\varepsilon_3 = 0, \quad 2)\ \mathrm{Im}\,(\varepsilon_1 - \varepsilon_3) = 0, \quad 3)\ \mathrm{Im}\,(\varepsilon_1 + \varepsilon_3) = 0. \tag{A.2}$$

It is also necessary to take into account the symmetry properties of $\gamma^{(1)}$ and γ relative to the change of sign of the frequency variables.

We now denote by $\gamma_1(\epsilon_1\epsilon_3)$ the value of the function γ on the upper edges of all the cuts (A.2) and by $\gamma_2(\epsilon_1\epsilon_3)$ the value of γ on the lower edge of the cut 2) and on the upper edges of the remaining cuts. The corresponding values of the function $\gamma^{(1)}$ are denoted by $\gamma^{(1)}_{1,2}$.

The equations for γ_1 and γ_2 are determined as the result of analytic continuation of (A.1) on the real values of the frequencies, and have the form

$$\begin{aligned}\gamma_1(\varepsilon_1\varepsilon_3) - \gamma_1^{(1)}(\varepsilon_1\varepsilon_3) \to \frac{-1}{2\pi i}\int_{-\infty}^{\infty} d\varepsilon\, \mathrm{th}\frac{\varepsilon}{2T} G_R(\varepsilon) G_A(-\varepsilon) \\ \times \gamma_1^{(1)}(\varepsilon_1\varepsilon)\gamma_2(\varepsilon\varepsilon_3) - \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\varepsilon\, \mathrm{cth}\frac{\varepsilon}{2T}\{G_R(\varepsilon+\varepsilon_3) G_A \\ \times(-\varepsilon-\varepsilon_3)\gamma_1^{(1)}(\varepsilon_1, \varepsilon+\varepsilon_3)[\gamma_1(\varepsilon+\varepsilon_3, \varepsilon_3) \\ - \gamma_2(\varepsilon+\varepsilon_3, \varepsilon_3)] + G_R(\varepsilon+\varepsilon_1) G_A(-\varepsilon-\varepsilon_1)\gamma_1(\varepsilon+\varepsilon_1\varepsilon_3) \\ \times[\gamma_2^{(1)}(\varepsilon_1, \varepsilon+\varepsilon_1) - \gamma_1^{(1)}(\varepsilon_1, \varepsilon+\varepsilon_1)]\},\end{aligned} \tag{A.3}$$

$$\begin{aligned}\gamma_2(\varepsilon_1\varepsilon_3) - \gamma_2^{(1)}(\varepsilon_1\varepsilon_3) \to \frac{-1}{2\pi i}\int_{-\infty}^{\infty} d\varepsilon\, \mathrm{th}\frac{\varepsilon}{2T} G_R(\varepsilon) G_A(-\varepsilon) \\ \times\gamma_1^{(1)}(\varepsilon_1\varepsilon)\gamma_2(\varepsilon\varepsilon_3) - \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\varepsilon\, \mathrm{cth}\frac{\varepsilon}{2T}\{G_R(\varepsilon+\varepsilon_s) G_A \\ \times(-\varepsilon-\varepsilon_3)\gamma_2^{(1)}(\varepsilon_1, \varepsilon+\varepsilon_3)[\gamma_1(\varepsilon+\varepsilon_3, \varepsilon_3) \\ - \gamma_2(\varepsilon+\varepsilon_3, \varepsilon_3)] + G_R(\varepsilon+\varepsilon_1) G_A(-\varepsilon-\varepsilon_1) \\ \times\gamma_2(\varepsilon+\varepsilon_1, \varepsilon_3)[\gamma_2^{(1)}(\varepsilon_1, \varepsilon+\varepsilon_1) - \gamma^{(1)}(\varepsilon_1, \varepsilon+\varepsilon_1)]\}.\end{aligned} \tag{A.4}*$$

Here G_R and G_A are respectively the retarded and advanced Green's functions of the electron at $T = T_0$. For brevity, the dependences on the momenta and the integration over them are omitted in (A.3), (A.4).

The functions $G_{R,A}$ and $\gamma^{(1)}$ do not depend on T. Terms with $\coth(\epsilon/2T)$ contain a factor of the form $(\gamma_1 - \gamma_2)$ which represents the difference of retarded and advanced functions of the Bose type, i.e., it vanishes for $\epsilon = 0$. For the same reason, at $T \ll \mu$, one can make the substitution:

$$\mathrm{cth}\frac{\varepsilon}{2T} \to \mathrm{sign}\,\varepsilon. \tag{A.5}$$

The justification of such a substitution is the fact that after it the equation for the difference $\gamma_1 - \gamma_2$ generally does not contain T, that is, $\gamma_1 - \gamma_2$ is not a singular function and therefore has the characteristic dimension of $\sim\mu$ in frequency.

After the substitution (A.5), the kernels of Eqs. (A.3) and (A.4) depend on T only through $\tanh(\epsilon/2T)$. This function differs from sign ϵ in the narrow region $\epsilon \sim T$. Taking this into account, we differentiate (A.3) with respect to T:

$$\begin{aligned}\frac{\partial\gamma_1(\varepsilon_1\varepsilon_3)}{\partial T} \to \frac{1}{4\pi i T^2}\int_{-\infty}^{\infty}\frac{\varepsilon\, d\varepsilon}{\mathrm{ch}^2(\varepsilon/2T)} G_R(\varepsilon) G_A(-\varepsilon)\gamma_1^{(1)}(\varepsilon_1\varepsilon)\gamma_2(\varepsilon\varepsilon_3) \\ - \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\varepsilon\, \mathrm{th}\frac{\varepsilon}{2T} G_R(\varepsilon) G_A(-\varepsilon)\gamma_1^{(1)}(\varepsilon_1\varepsilon)\frac{\partial\gamma_1(\varepsilon\varepsilon_3)}{\partial T} \\ - \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\varepsilon\, \mathrm{sign}(\varepsilon-\varepsilon_1) G_R(\varepsilon) G_A(-\varepsilon)[\gamma_2^{(1)}(\varepsilon_1\varepsilon) \\ - \gamma_1^{(1)}(\varepsilon_1\varepsilon)]\frac{\partial\gamma_1(\varepsilon\varepsilon_3)}{\partial T}.\end{aligned} \tag{A.6}$$

Here it is taken into account that $\partial\gamma_1/\partial T = \partial\gamma_2/\partial T$.

Integration in the first term on the right in (A.6) is carried out in the narrow region $\epsilon, v|p - p_0| \sim T$. In this region γ and $\gamma^{(1)}$ change slightly and one can take them out from under the integral sign. Then the integral is computed if one makes use of the expression for G (1.4). As a result, Eq. (A.6) takes the form

$$\begin{aligned}\frac{\partial\gamma_1(\varepsilon_1\varepsilon_3)}{\partial T} - \frac{\alpha}{T}\gamma_1^{(1)}(\varepsilon_1 0)\gamma_2(0\varepsilon_3) \\ \to \frac{-1}{2\pi i}\int_{-\infty}^{\infty} d\varepsilon\, \mathrm{th}\frac{\varepsilon}{2T} G_R(\varepsilon) G_A(-\varepsilon)\gamma_1^{(1)}(\varepsilon_1\varepsilon) \\ \times\frac{\partial\gamma_1(\varepsilon\varepsilon_3)}{\partial T} - \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\varepsilon\, \mathrm{sign}(\varepsilon-\varepsilon_1) G_R(\varepsilon) G_A(-\varepsilon)[\gamma_2^{(1)}(\varepsilon_1\varepsilon) \\ - \gamma_1^{(1)}(\varepsilon_1\varepsilon)]\frac{\partial\gamma_1(\varepsilon\varepsilon_3)}{\partial T};\end{aligned}$$

$$\alpha = p_0^2 a_0^2 / 2\pi^2 v_0, \qquad a_0 = a|_{T=T_0}, \qquad v_0 = v|_{T=T_0}. \tag{A.7}$$

The solution of Eq. (A.7) has the form

$$\partial\gamma_1(p_1p_3)/\partial T = \alpha T^{-1}\gamma_1(p_1 0)\gamma_2(0p_3), \tag{A.8}$$

which can be established by direct substitution. In (A.8) p_1 and p_3 are 4-momenta and

$$\gamma_1(p_1 0) \equiv \gamma_1(p_1p_3)|_{\varepsilon=|p_3|-p_0=0}.$$

For $\gamma_1(00)$ the equation has the form

$$\partial\gamma_1(00)/\partial T = \alpha T^{-1}\gamma_1^2(00).$$

*th = tanh, cth = coth.

The solution of this equation, which goes to infinity for $T = T_0$, has the form $-\gamma_1^{-1}(00) = \alpha \ln(T/T_0)$. For $T - T_0 \ll T_0$, we get

$$\gamma_1(00) = -\frac{T_0}{\alpha}\frac{1}{T - T_0}. \quad \text{(A.9)}$$

For $\gamma(p_1p_3)$, we get the expression

$$\gamma(p_1p_3) = -\chi(p_1)\chi(p_3) / (T - T_0), \quad \text{(A.10)}$$

where the function $\chi(p)$ obeys the equation

$$\chi(p) = -T_0 \sum_n \int \frac{d^3k}{(2\pi)^3} \gamma^{(1)}(pk) G(k) G(-k) \chi(k) \quad \text{(A.11)}$$

and is normalized by the condition

$$\chi(p)|_{\varepsilon=|p|-p_0=0} = (T_0/\alpha)^{1/2}. \quad \text{(A.12)}$$

Equation (A.11) is obtained if one substitutes (A.10) and (A.1) and takes the limit $T_k \to T_0$. The equation (A.10) is valid if the electrons are combined with zero total momentum.

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