WAVE EQUATIONS WITH ZERO AND NONZERO REST MASSES

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Submitted to JETP editor February 27, 1959

J. Exptl. Theoret. Phys. (U.S.S.R.) **37**, 470-476 (August, 1959)

It is shown that wave equations with nonzero rest mass are invariant with respect to a 15-parameter group of transformations which is a representation of the conformal group.

1. INTRODUCTION

THE Klein-Gordon and Dirac equations are invariant with respect to a 10-parameter group of transformations which is a representation of the inhomogeneous Lorentz group L_{10}.

It has been shown by Cunnigham,[1] Bateman,[2] Dirac,[3] Bhabha,[4] Pauli,[5] McLennan,[6] and others that in the case of zero rest mass the wave equations are invariant under a wider, 15-parameter group of transformations which forms a representation of the conformal group C_4 including L_{10}. The Dirac equation for the neutrino is in addition invariant under the 4-parameter Pauli group.

It is generally believed that these invariance properties are peculiar to wave equations with zero rest mass.

However, we show below that both the Klein-Gordon and the Dirac equations with nonzero rest mass are also invariant under a 15-parameter group of transformations G_{15} which is a representation of the conformal group C_4. The analogue of Pauli's group also exists for the Dirac equation. The operators for all these transformations contain as a parameter the mass m, and in the limit $m = 0$ go over into the well known operators.

Some of the operators of the representation of the Lorentz group L_{10} assume an unusual form in the group G_{15}. This causes certain difficulties: under Lorentz rotations the momentum of a particle does not transform as a four-vector.

In the derivation of the transformations for nonzero rest mass we shall made use of the known form of the transformation for $m = 0$.

In this connection we discuss in Sec. 2 the representation of the conformal group C_4 for rest mass zero. In Sec. 3 we establish the method for deriving the corresponding transformations for $m \neq 0$. In Sec. 4 we give the infinitesimal operators of the G_{15} for wave equations with $m \neq 0$ and the analogue of the Pauli group.

2. CONFORMAL GROUP AND WAVE EQUATIONS WITH ZERO REST MASS

As indicated in the introduction, many papers exist[1-6] dealing with the proof of conformal invariance of wave equations with zero rest mass. In this section we give a summary of the principal results, some of which appear in references 3 — 8, relevant to the problem of the conformal group and the invariance of the Klein-Gordon and Dirac equations with zero rest mass. The 15-parameter conformal group C_4 consists of the Lorentz group L_{10} (translations and rotations), of the scale transformation, and of the four properly conformal transformations (see first line, Table I). Properly conformal transformations correspond to the product of an inversion in the unit hyper-sphere $x^1 = x_\mu / x^2$, then a translation, and then again an inversion. Three of them (spatial) are related to transitions to a uniformly accelerating frame of reference.[9-13]

The transformation laws for solutions of the Klein-Gordon and Dirac equations with rest mass zero

$$\Box^2 \varphi_0 (x) = 0, \tag{1}$$

$$\gamma_\mu \frac{\partial}{\partial x_\mu} \psi_0 (x) = 0. \tag{2}$$

are given in lines 3 and 4 of Table I.

As can be seen from Table I the solution of even the Klein-Gordon equation does not behave like a scalar under scale transformations and properly conformal transformations.

Functions transforming according to any of the representations of the conformal group change according to the following law under infinitesimal transformations:*

$$\psi' (x) = (1 + i a_\mu P_\mu) \psi (x), \tag{3}$$

*We take the physical momentum and angular momentum operators for P_μ and $M_{\mu\nu}$ – hence the appearance of the imaginary unit i.

TABLE I. Finite transformations

	Scale transformation (1 parameter)*	Properly conformal transformations (4 parameters)**
Transformation of coordinates	$x'_\mu = \frac{x_\mu}{a}$	$x'_\mu = \frac{x_\mu - \alpha_\mu x^2}{1-2(\alpha x) + \alpha^2 x^2}$
Transformation of a scalar function	$f'(x_\mu) = f(ax_\mu)$	$f'(x_\mu) = f\left(\frac{x_\mu + \alpha_\mu x^2}{1+2(\alpha x) + \alpha^2 x^2}\right)$
Transformation of solutions of the Klein-Gordon equation with m = 0	$\varphi'_0(x_\mu) = a\varphi_0(ax_\mu)$	$\varphi'_0(x_\mu) = [1+2(\alpha x) + \alpha^2 x^2]^{-1} \varphi_0\left(\frac{x_\mu + \alpha_\mu x^2}{1+2(\alpha x)+\alpha^2 x^2}\right)$
Transformation of solutions of the Dirac equation with m = 0.	$\psi'_0(x_\mu) = a^{3/2} \psi_0(ax_\mu)$	$\psi'_0(x_\mu) = \frac{1+2(\alpha x) - (\alpha\gamma)(\gamma x)}{[1+2(\alpha x) + \alpha^2 x^2]^2} \psi_0\left(\frac{x_\mu + \alpha_\mu x^2}{1+2(\alpha x)+\alpha^2 x^2}\right)$

*The numerical value of the power to which a must be raised in the case of $\phi_0(ax)$ and $\psi_0(ax)$ is determined by the requirement that ϕ_0 and ψ_0 transform according to a representation of the group C_4 in its entirety, and may be found from the structural relation (15).

**$x^2 = x_\mu x_\mu$, $\alpha^2 = \alpha_\mu \alpha_\mu$, $(\alpha x) = \alpha_\mu x_\mu$, $\mu = 1,2,3,4$.

$$\psi'(x) = (1 + i\omega_{\mu\nu} M_{\mu\nu}) \psi(x), \tag{4}$$

$$\psi'(x) = (1 + \varepsilon I) \psi(x), \tag{5}$$

$$\psi'(x) = (1 + \alpha_\mu I_\mu) \psi(x), \tag{6}$$

where P_μ, $M_{\mu\nu}$, I, and I_μ are the infinitesimal operators for the translation, rotation, scale and properly conformal transformations, and a_μ, $\omega_{\mu\nu}$, ϵ and α_μ are the corresponding infinitesimal transformation parameters.

The infinitesimal operators of the conformal group and its representations satisfy the following structural relations:

$$[P_\mu, P_\nu] = 0, \tag{7}$$

$$[P_\mu, M_{\nu\lambda}] = -i(\delta_{\mu\nu} P_\lambda - \delta_{\mu\lambda} P_\nu), \tag{8}$$

$$[M_{\mu\nu}, M_{\lambda\rho}] = i(\delta_{\mu\lambda} M_{\nu\rho} + \delta_{\nu\rho} M_{\mu\lambda} - \delta_{\mu\rho} M_{\nu\lambda} - \delta_{\nu\lambda} M_{\mu\rho}), \tag{9}$$

$$[I_\mu, M_{\nu\lambda}] = -i(\delta_{\mu\nu} I_\lambda - \delta_{\mu\lambda} I_\nu), \tag{10}$$

$$[I_\mu, I_\nu] = 0, \tag{11}$$

$$[P_\mu, I] = P_\mu, \tag{12}$$

$$[M_{\mu\nu}, I] = 0, \tag{13}$$

$$[I_\mu, I] = -I_\mu, \tag{14}$$

$$[P_\mu, I_\nu] = 2(M_{\mu\nu} + i\delta_{\mu\nu} I). \tag{15}$$

For solutions of the Klein-Gordon equation with zero mass the infinitesimal operators in x- and p-representations respectively $[\psi(p) = (2\pi)^{-2} \times \int \exp(-ipx) \psi(x) dx]$ are given as follows

$$\begin{aligned}
&P_\mu = -i\partial/\partial x_\mu, \quad P_\mu = p_\mu, \\
&M_{\mu\nu} = -i(x_\mu \partial/\partial x_\nu - x_\nu \partial/\partial x_\mu), \\
&M_{\mu\nu} = -i(p_\mu \partial/\partial p_\nu - p_\nu \partial/\partial p_\mu), \\
&I = x_\mu \partial/\partial x_\mu + 1, \quad I = -p_\mu \partial/\partial p_\mu - 3, \\
&I_\mu = x^2 \partial/\partial x_\mu - 2x_\mu x_\nu \partial/\partial x_\nu - 2x_\mu, \\
&I_\mu = -i\{p_\mu \partial^2/\partial p_\nu^2 - 2[3 + p_\nu \partial/\partial p_\nu] \partial/\partial p_\mu\}.
\end{aligned} \tag{16}$$

For solutions of the Dirac equation with m = 0 the corresponding quantities are

$$\begin{aligned}
&P_\mu = -i\partial/\partial x_\mu, \quad P_\mu = p_\mu, \\
&M_{\mu\nu} = -i(x_\mu \partial/\partial x_\nu - x_\nu \partial/\partial x_\mu) + \tfrac{1}{2}\sigma_{\mu\nu}, \\
&M_{\mu\nu} = -i(p_\mu \partial/\partial p_\nu - p_\nu \partial/\partial p_\mu) + \tfrac{1}{2}\sigma_{\mu\nu}, \\
&I = x_\mu \partial/\partial x_\mu + {}^3/_2, \quad I = -p_\mu \partial/\partial p_\mu - {}^5/_2, \\
&I_\mu = x^2 \partial/\partial x_\mu - 2x_\mu x_\nu \partial/\partial x_\nu - 2x_\mu - \gamma_\mu(\gamma x), \\
&I_\mu = -i(p_\mu \partial^2/\partial p_\nu^2 - 2(3 + p_\nu \partial/\partial p_\nu)\partial/\partial p_\mu + \gamma_\mu \gamma_\nu \partial/\partial p_\nu),
\end{aligned} \tag{17}$$

where $\sigma_{\mu\nu} = -i(\gamma_\mu \gamma_\nu - \delta_{\mu\nu})$.

Whereas the operators P_μ, $M_{\mu\nu}$ commute with the wave equation operators, the corresponding commutators for the operators I and I_μ are (in the p-representation): for the Klein-Gordon equation

$$[I, p^2] = -2p^2, \quad [I_\mu, p^2] = 4i\partial p^2/\partial p_\mu, \tag{18}$$

and for the Dirac equation

$$[I, i\gamma p] = -i\gamma p, \quad [I_\mu, i\gamma p] = 2i\partial(i\gamma p)/\partial p_\mu. \tag{19}$$

Consequently these commutators vanish when applied to solutions of the wave equations and therefore the transformed functions will also be solutions of the wave equations.

If instead of I_μ, P_μ, and I one introduces the operators

$$M_{\mu 5} = \frac{1}{2}(I_\mu + iP_\mu), \qquad M_{\mu 6} = \frac{1}{2}(P_\mu + iI_\mu),$$
$$M_{56} = -I \qquad (\mu \neq 5,6), \tag{20}$$

then the relations (7) — (15) may be written in the form of structural relations for the rotation group in six dimensions, i.e., in the form (9) with μ, ν, λ, ρ = 1, 2, 3, 4, 5, 6. A number of authors[3,4,14] have studied the conformal group and invariance with respect to it of wave equations with zero rest mass by going over to a 6-dimensional space.

3. RELATION BETWEEN WAVE EQUATIONS WITH m ≠ 0 AND m = 0

The proof of invariance of the Klein-Gordon and Dirac equations with mass under the 15-parameter group will be accomplished by establishing a relation between the equations with and without mass. All considerations will be carried out in momentum space.

If in the Klein-Gordon equation

$$[\mathbf{p}^2 - p_0^2 + m^2]\,\varphi(\mathbf{p}, p_0) = 0 \tag{21}$$

one makes the substitution

$$\mathbf{q} = \mathbf{p}, \qquad q_0 = \varepsilon(p_0)\sqrt{p_0^2 - m^2}, \tag{22}$$
$$\varphi_0(\mathbf{q}, q_0) = \varphi(\mathbf{p}, p_0), \tag{23}$$

then Eq. (21) takes on the form of a Klein-Gordon equation with zero mass

$$(\mathbf{q}^2 - q_0^2)\,\varphi_0(\mathbf{q}, q_0) = 0. \tag{24}$$

Similarly, the Dirac equation

$$(i\gamma p + m)\,\psi(\mathbf{p}, p_0) = 0 \tag{25}$$

goes over, under the substitution (22) and

$$\psi_0(\mathbf{q}, q_0) = S\psi(\mathbf{p}, p_0), \tag{26}$$

where

$$S = \cosh\frac{\chi}{2} - \gamma_4 \sinh\frac{\chi}{2}, \quad \chi = \tanh^{-1}\frac{m}{p_0}, \tag{27}$$

into the Dirac equation for zero mass

$$i\gamma q\psi_0(\mathbf{q}, q_0) = 0, \tag{28}$$

since

$$S^{-1}(i\gamma p + m)\,S^{-1} = i\gamma q. \tag{29}$$

There exists a whole class of transformations of this kind which reduce the Klein-Gordon and Dirac equations to mass m = 0. They are all non-covariant. One of them, with a unitary S matrix, was utilized by Cini and Touschek.[15]

Equations (25) and (28) are invariant with respect to the 15-parameter conformal group. The laws of transformation for the solutions of these equations are given in Sec. 2. The 15-parameter transformation group G_{15} which leaves the equation with nonzero mass invariant may be obtained as follows:

1) Using (22) and (23) for the Klein-Gordon equation, or (22) and (26) for the Dirac equation, we perform the reduction to the massless equations.

2) We perform any of the transformations of the 15-parameter group for m = 0.

3) Finally we perform the operations inverse to (22) and (23) or (22) and (26) and thus obtain the operators of the 15-parameter group for nonzero mass.

4. INFINITESIMAL OPERATORS OF THE 15-PARAMETER GROUP FOR KLEIN-GORDON AND DIRAC EQUATIONS WITH m ≠ 0.

Finite transformation of the 15-parameter group G_{15} for m ≠ 0 are very complicated. However, since they are fully determined by their corresponding infinitesimal operators we give below only the latter, derived in the manner described in the previous section.

The infinitesimal operators are as follows:*

a) For the Klein-Gordon equation

$$P_r^K = p_r \ (r = 1, 2, 3), \qquad P_0^K = \sqrt{p_0^2 - m^2}; \tag{30}$$

$$M_{rn}^K = \frac{1}{i}\left(p_r\frac{\partial}{\partial p_n} - p_n\frac{\partial}{\partial p_r}\right),$$

$$M_{r4}^K = \frac{1}{i}\frac{\sqrt{p_0^2 - m^2}}{p_0}\left(p_r\frac{\partial}{\partial p_4} - p_4\frac{\partial}{\partial p_r}\right) \qquad (p_4 = ip_0); \tag{31}$$

$$I^K = -\left[3 + p_\mu\frac{\partial}{\partial p_\mu} - \frac{m^2}{p_0}\frac{\partial}{\partial p_0}\right]; \tag{32}$$

$$I_r^K = -ip_r\left[\frac{\partial^2}{\partial p_\mu^2} + \frac{m^2}{p_0^2}\frac{\partial^2}{\partial p_0^2} - \frac{m^2}{p_0^3}\frac{\partial}{\partial p_0}\right] - 2iI^K\frac{\partial}{\partial p_r},$$

$$I_0^K = -i\frac{\sqrt{p_0^2 - m^2}}{p_0}\left\{p_0\left[\frac{\partial^2}{\partial p_\mu^2} + \frac{m^2}{p_0^2}\frac{\partial^2}{\partial p_0^2} + \frac{m^2}{p_0^3}\frac{\partial}{\partial p_0}\right]\right.$$
$$\left. - 2I^K\frac{\partial}{\partial p_0}\right\}. \tag{33}$$

*For the sake of simplicity we consider only the case of positive frequencies, $p_0 \geqslant m$.

b) For the Dirac equation

$$P_r^D = p_r, \qquad P_0^D = \sqrt{p_0^2 - m^2}; \tag{34}$$

$$M_{rn}^D = M_{rn}^K + {}^1/_2\sigma_{rn},$$

$$M_{r4}^D = M_{r4}^K + \frac{\sqrt{p_0^2 - m^2}}{2p_0}\sigma_{r4} + \frac{m}{2p_0\sqrt{p_0^2 - m^2}}(\gamma_r p_4 - \gamma_4 p_r + m\sigma_{r4}); \tag{35}$$

$$I^D = -\left[\frac{5}{2} + p_\mu\frac{\partial}{\partial p_\mu} - \frac{m^2}{p_0}\frac{\partial}{\partial p_0} + \gamma_4\frac{m}{2p_0}\right]; \tag{36}$$

$$I_r^D = I_r^K + \frac{im}{4p_0^3(p_0^2 - m^2)}[mp_0p_r - 2p_r\gamma_4(2p_0^2 - m^2) + 2i\gamma_r p_0^2(p_0 - \gamma_4 m)] - i\gamma_r\left(\gamma_\mu\frac{\partial}{\partial p_\mu} + \frac{im}{p_0}\frac{\partial}{\partial p_0}\right) + \frac{im\gamma_4}{p_0^2}\left(p_r\frac{\partial}{\partial p_0} + p_0\frac{\partial}{\partial p_r}\right),$$

$$I_0^D = I_0^K + \frac{im}{4p_0^3\sqrt{p_0^2 - m^2}}[2\gamma_4(2p_0^2 - m^2) + mp_0] + \frac{im\gamma_4}{p_0\sqrt{p_0^2 - m^2}}I^D - \frac{m + \gamma_4 p_0}{\sqrt{p_0^2 - m^2}}\left(\gamma_\mu\frac{\partial}{\partial p_\mu} + \frac{im}{p_0}\frac{\partial}{\partial p_0}\right). \tag{37}$$

The infinitesimal operators (30) — (33) and (34) — (37) satisfy the structural relations (7) — (15). Consequently the transformations determined by them form a representation of the conformal group C_4. In the limit $m = 0$ they go over into the corresponding operators for the massless equations [see Eqs. (16) and (17)].

All these infinitesimal operators either commute with the appropriate wave equation operator, or commute in application to the solutions of these equations, in a manner similar to the $m = 0$ case [see Eqs. (18) and (19)], e.g.,

$$[I^D, i\gamma p + m] = -(1 + \gamma_4 m/p_0)(i\gamma p + m). \tag{38}$$

Let us note that all commutators that vanish for the $m = 0$ case also vanish for $m \neq 0$ with the exception of

$$[M_{r4}^D, i\gamma p + m] = \frac{m}{p_0\sqrt{p_0^2 - m^2}}(p_r\gamma_4 - p_4\gamma_r)(i\gamma p + m). \tag{39}$$

Thus, the Klein-Gordon and Dirac equations with mass are invariant with respect to the 15-parameter group G_{15}.

However in order to obtain a representation of the conformal group C_4 in its entirety it was necessary to modify the representation of the inhomogeneous Lorentz group L_{10} (the operators P_0 and M_{r4} were changed). As a consequence the law of transformation for the four-momentum of a particle p_μ (which is not the same, in general, as the operator of infinitesimal translation P_μ) will differ from the transformation law obeyed by four-vectors.* Analogous difficulties may also arise for other physical quantities.

In conclusion we note that the method described in Sec. 2 may be used to derive the analogue of the Pauli group for the Dirac equation with mass. For example, the one-parameter group is given by

$$\psi'(p) = \exp(i\alpha\Gamma_5)\psi(p), \tag{40}$$

where

$$\Gamma_5 = \varepsilon(p_0)\frac{p_0 + \gamma_4 m}{\sqrt{p_0^2 - m^2}}\gamma_5, \qquad \Gamma_5^2 = 1. \tag{41}$$

In the representation discussed above of the conformal group C_4 the quantity Γ_5 behaves like a Lorentz pseudoscalar.

The authors are grateful to Prof. M. A. Markov for his interest in this research and to L. G. Zastavenko for useful discussions.

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Translated by A. M. Bincer
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*p_μ transforms as a four-vector.